Stochastic Calculus

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Contents

Chapter 1

Stochastic integral

1.1 Quadratic variation

Definition 1.1.1. A continuous martingale $X = \{X_t : t \geq 0\}$ (w.r.t $\mathcal{F}_t$) is square-integrable if for any $t \geq 0$,

$$\mathbb{E}(X_t^2) < \infty.$$

Let $\mathcal{M}_2$ denote the space of all square-integrable, $\mathbb{P}$-almost surely continuous martingales with $X_0 = 0$.

Definition 1.1.2. If $X \in \mathcal{M}_2$, then $X^2 = \{X_t^2 : t \geq 0\}$ is a non-negative submartingale, thus by Doob-Meyer decomposition theorem, there is a unique adapted, predictable increasing process A_t, s.t. $A_0 = 0$ a.s. and $X^2 - A_t$ is a martingale. We call such A_t the quadratic variation of X, denoted as $[X](t)$.

Remark. 1.Ignoring the detailed proof, the continuity of X implies the continuity of $[X](t)$.
2. For any $a \in \mathbb{R}$, $[aX](t) = a^2[X](t)$.
3. The reason that we call it quadratic variation can be seen in Theorem 1.1.5.

Example 1.1.3. Let $B = \{B_t : t \geq 0\}$ be a BM, then $B \in \mathcal{M}_2$ since

$$\mathbb{E}(B_t^2) = t < \infty.$$

And we can show $B_t^2 - t$ is a martingale, thus the quadratic variation of B is $[B](t) = t$.

Definition 1.1.4. Let $X, Y \in \mathcal{M}_2$, define their covariation (or called cross-variation) $[X, Y]$

by

$$[X,Y]_t := \frac{1}{4}([X+Y]_t - [X-Y]_t).$$

Remark. 1. The definition is an analog of the polarization identity.
2. $[X,X](t) = [X]_t$.

Theorem 1.1.5. *Suppose $X \in \mathcal{M}_2$, Π is a partition of $[0,t]$, then as $|\Pi| \to 0$,*

$$V_t^{(2)}(\Pi) := V^{(2)}(X, \Pi, [0,t]) \to [X](t), \quad \textit{in probability.}$$

Lemma 1.1.6. *Suppose $X \in \mathcal{M}_2$ satisfies*

$$\sup_{s\in[0,t]} |X_s| \le K < \infty, \quad a.s.,$$

and $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of $[0,t]$, then

$$\mathbb{E}[(V_t^{(2)}(\Pi))^2] \le 6K^2.$$

Proof. By definition,

$$V_t^{(2)}(\Pi) = \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^2,$$

so

$$\begin{aligned}\mathbb{E}[(V_t^{(2)}(\Pi))^2] &= \mathbb{E}\left[\left(\sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^2\right)^2\right] \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left[|X_{t_{k+1}} - X_{t_k}|^4\right] + 2\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1} \mathbb{E}\left[|X_{t_{j+1}} - X_{t_j}|^2 |X_{t_{k+1}} - X_{t_k}|^2\right].\end{aligned}$$

By the properties of martingales,

$$\begin{aligned}\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2|X_{t_{k+1}}-X_{t_k}|^2\right] &= \sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}\mathbb{E}\left[\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2|X_{t_{k+1}}-X_{t_k}|^2\middle|\mathcal{F}_{t_{j+1}}\right]\right]\\ &= \sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2\mathbb{E}\left[|X_{t_{k+1}}-X_{t_k}|^2\middle|\mathcal{F}_{t_{j+1}}\right]\right]\\ &= \sum_{j=0}^{n-2}\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2\sum_{k=j+1}^{n-1}\mathbb{E}\left[|X_{t_{k+1}}-X_{t_k}|^2\middle|\mathcal{F}_{t_{j+1}}\right]\right]\end{aligned}$$

Notice that for fixed j,

$$\begin{aligned}\sum_{k=j+1}^{n-1}\mathbb{E}\left[|X_{t_{k+1}}-X_{t_k}|^2\middle|\mathcal{F}_{t_{j+1}}\right] &= \sum_{k=j+1}^{n-1}\mathbb{E}\left[X_{t_{k+1}}^2+X_{t_k}^2-2X_{t_k}X_{t_{k+1}}\middle|\mathcal{F}_{t_{j+1}}\right]\\ &= \sum_{k=j+1}^{n-1}\mathbb{E}\left[X_{t_{k+1}}^2+X_{t_k}^2-2\mathbb{E}\left[X_{t_k}X_{t_{k+1}}|\mathcal{F}_{t_k}\right]\middle|\mathcal{F}_{t_{j+1}}\right]\\ &= \sum_{k=j+1}^{n-1}\mathbb{E}\left[X_{t_{k+1}}^2+X_{t_k}^2-2X_{t_k}\mathbb{E}\left[X_{t_{k+1}}|\mathcal{F}_{t_k}\right]\middle|\mathcal{F}_{t_{j+1}}\right]\\ &= \sum_{k=j+1}^{n-1}\mathbb{E}\left[X_{t_{k+1}}^2+X_{t_k}^2-2X_{t_k}^2\middle|\mathcal{F}_{t_{j+1}}\right]\\ &= \mathbb{E}[X_n^2-X_{j+1}^2|\mathcal{F}_{t_{j+1}}]\le\mathbb{E}[X_n^2|\mathcal{F}_{t_{j+1}}]\le K^2.\end{aligned}$$

therefore

$$\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2|X_{t_{k+1}}-X_{t_k}|^2\right]\le K^2\sum_{j=0}^{n-2}\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2\right]\le K^4,$$

where

$$\sum_{j=0}^{n-2}\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2\right]=\sum_{j=0}^{n-2}\mathbb{E}\left[\mathbb{E}\left[|X_{t_{j+1}}-X_{t_j}|^2\middle|\mathcal{F}_{t_0}\right]\right]\le K^2.$$

And we also have

$$
\begin{aligned}
\sum_{k=0}^{n-1}\mathbb{E}\left[|X_{t_{k+1}}-X_{t_k}|^4\right] &\le \sum_{k=0}^{n-1}\mathbb{E}\left[\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|^2\}\cdot|X_{t_{k+1}}-X_{t_k}|^2\right]\\
&=\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|^2\}\sum_{k=0}^{n-1}\mathbb{E}\left[|X_{t_{k+1}}-X_{t_k}|^2\right]\\
&\le 4K^2\sum_{k=0}^{n-1}\mathbb{E}\left[|X_{t_{k+1}}-X_{t_k}|^2\right]\\
&\le 4K^4,
\end{aligned}
$$

where

$$\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|^2\}\le \max_{0\le k\le n-1}\{2X_{t_{k+1}}^2+2X_{t_k}^2\}\le 4K^2.$$

Combining together, we have

$$\mathbb{E}[(V_t^{(2)}(\Pi))^2]\le 4K^4+2K^4=6K^4.$$

□

Lemma 1.1.7. *Suppose $X\in\mathcal{M}_2$ satisfies*

$$\sup_{s\in[0,t]}|X_s|\le K<\infty,\quad a.s.$$

For partitions Π of $[0,t]$, we have

$$\lim_{|\Pi|\to 0}\mathbb{E}[V_t^{(4)}(\Pi)]=0.$$

Proof. For the partition $\Pi=\{0=t_0<t_1<\cdots<t_n=t\}$, we have

$$
\begin{aligned}
V_t^{(4)}(\Pi)=\sum_{k=0}^{n-1}|X_{t_{k+1}}-X_{t_k}|^4 &\le \max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|^2\}\sum_{k=0}^{n-1}|X_{t_{k+1}}-X_{t_k}|^2\\
&=\left(\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\right)^2 V_t^{(2)}(\Pi),
\end{aligned}
$$

so by Cauchy-Schwarz and Lemma 1.1.6

$$\begin{aligned}\mathbb{E}[V_t^{(4)}(\Pi)] &= \mathbb{E}\left[\left(\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\right)^2 V_t^{(2)}(\Pi)\right]\\ &\le \sqrt{\mathbb{E}\left[\left(\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\right)^4\right]\mathbb{E}\left[(V_t^{(2)}(\Pi))^2\right]}\\ &\le \sqrt{6}K^2\sqrt{\mathbb{E}\left[\left(\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\right)^4\right]}.\end{aligned}$$

Claim:As $|\Pi|\to 0$,

$$\mathbb{E}\left[\left(\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\right)^4\right]\to 0.$$

Since X_s is uniformly continuous on $[0,t]$ a.s., then

$$\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\overset{|\Pi|\to 0}{\longrightarrow} 0,\quad a.s.$$

And

$$\max_{0\le k\le n-1}\{|X_{t_{k+1}}-X_{t_k}|\}\le 4K^2,$$

then bounded convergence theorem implies that the claim holds. □

Proof of Theorem 1.1.5. We will prove a special case: suppose $\sup_{s\in[0,t]}|X_s|\le K<\infty$ a.s.. Then

$$\mathbb{E}([X]_s)=\mathbb{E}[X_s^2]\le K^2<\infty,\quad \forall s\in[0,t].$$

For any partition $\Pi=\{0=t_0<t_1<\cdots<t_n=t\}$, we have

$$\begin{aligned}\mathbb{E}\left[\left(V_t^{(2)}(\Pi)-[X]_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{k=0}^{n-1}|X_{k+1}-X_k|^2-([X]_{t_{k+1}}-[X]_{t_k})\right)^2\right]\\ &= \sum_{k=0}^{n-1}\mathbb{E}(a_k^2)+2\sum_{0\le j<k\le n-1}\mathbb{E}(a_j a_k),\end{aligned}$$

where $a_k=|X_{t_{k+1}}-X_{t_k}|^2-([X]_{t_{k+1}}-[X]_{t_k})$.

If $j < k$, then $t_j < t_{j+1} \leq t_k < t_{k+1}$,

$$\mathbb{E}(a_j a_k) = \mathbb{E}(\mathbb{E}(a_j a_k|\mathcal{F}_{t_k})) = \mathbb{E}(a_j \mathbb{E}(a_k|\mathcal{F}_{t_k})),$$

in which

$$\begin{aligned}
\mathbb{E}(a_k|\mathcal{F}_{t_k}) &= \mathbb{E}\left[|X_{t_{k+1}} - X_{t_k}|^2 - ([X]_{t_{k+1}} - [X]_{t_k})\middle|\mathcal{F}_{t_k}\right] \\
&= \mathbb{E}\left[X_{t_{k+1}}^2 + X_{t_k}^2 - 2X_{t_k}X_{t_{k+1}} - ([X]_{t_{k+1}} - [X]_{t_k})\middle|\mathcal{F}_{t_k}\right] \\
&= \mathbb{E}\left[X_{t_{k+1}}^2 + X_{t_k}^2 - 2X_{t_k}\mathbb{E}(X_{t_{k+1}}|\mathcal{F}_{t_k}) - ([X]_{t_{k+1}} - [X]_{t_k})\middle|\mathcal{F}_{t_k}\right] \\
&= \mathbb{E}\left[X_{t_{k+1}}^2 + X_{t_k}^2 - 2X_{t_k}^2 - ([X]_{t_{k+1}} - [X]_{t_k})\middle|\mathcal{F}_{t_k}\right] \\
&= \mathbb{E}\left[(X_{t_{k+1}}^2 - [X]_{t_{k+1}}) - (X_{t_k}^2 - [X]_{t_k})\middle|\mathcal{F}_{t_k}\right] = 0,
\end{aligned}$$

the last equality holds because $X_s^2 - [X]_s$ is a martingale by definition. Therefore $\mathbb{E}(a_j a_k) = 0$. Then

$$\begin{aligned}
\mathbb{E}\left[\left(V_t^{(2)}(\Pi) - [X]_t\right)^2\right] &= \sum_{k=0}^{n-1} \mathbb{E}\left[\left(|X_{t_{k+1}} - X_{t_k}|^2 - ([X]_{t_{k+1}} - [X]_{t_k})\right)^2\right] \\
&\leq 2\sum_{k=0}^{n-1} \mathbb{E}\left[|X_{t_{k+1}} - X_{t_k}|^4 + ([X]_{t_{k+1}} - [X]_{t_k})^2\right] \\
&= 2\mathbb{E}[V_t^{(4)}(\Pi)] + 2\sum_{k=0}^{n-1} \mathbb{E}\left[([X]_{t_{k+1}} - [X]_{t_k})^2\right] \\
&\leq \mathbb{E}[V_t^{(4)}(\Pi)] + 2\sum_{k=0}^{n-1} \mathbb{E}\left[([X]_{t_{k+1}} - [X]_{t_k}) \cdot \max_{0\leq k\leq n-1}\{[X]_{t_{k+1}} - [X]_{t_k}\}\right] \\
&\qquad \text{(since } [X]_s \text{ is increasing)} \\
&= \mathbb{E}[V_t^{(4)}(\Pi)] + 2\max_{0\leq k\leq n-1}\{[X]_{t_{k+1}} - [X]_{t_k}\} \cdot \mathbb{E}([X]_t) \\
&\leq \mathbb{E}[V_t^{(4)}(\Pi)] + 2K^2 \max_{0\leq k\leq n-1}\{[X]_{t_{k+1}} - [X]_{t_k}\} \to 0, \quad \text{as } |\Pi| \to 0
\end{aligned}$$

because $\mathbb{E}[V_t^{(4)}(\Pi)] \to 0$ by Lemma 1.1.7 and

$$\max_{0\le k\le n-1}\{[X]_{t_{k+1}} - [X]_{t_k}\} \to 0$$

by the uniform continuity of $[X]_s$ on $[0,t]$. Therefore we have shown

$$V_t^{(2)}(\Pi) \to [X]_t \quad \text{in } L^2,$$

hence also in probability.

1.2 Definition and properties of Itô integral

Definition 1.2.1. Let $\{B_t : t \ge 0\}$ be a Brownian motion (BM) defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_t : t \ge 0\}$ be a filtration s.t. B_t adapts to it. Fix $T > 0$.

(1) Define the space of adapted processes by

$$\mathcal{L}_A^2 = \mathcal{L}_A^2([0,T]\times\Omega) = \left\{ X : [0,T]\times\Omega \to \mathbb{R} \,\middle|\, \mathbb{E}\left[\int_0^T |X_s(\omega)|^2 \,\mathrm{d}s\right] < \infty,\ X(t,\omega)\in\mathcal{F}_t,\ \forall t \right\},$$

and define the space of simple adapted processes by

$$\mathcal{L}_{A,0}^2 = \mathcal{L}_{A,0}^2([0,T]\times\Omega) = \Bigg\{ X \in \mathcal{L}_A^2 : X_s(\omega) = c_0(\omega)\mathbb{1}_{\{0\}}(s) + \sum_{k=0}^{n-1} c_k(\omega)\mathbb{1}_{(t_k,t_{k+1}]}(s),$$
$$\text{for some partition } 0 = t_0 < t_1 < \cdots < t_n = T,$$
$$\text{and } c_k \in \mathcal{F}_{t_k} \text{ for all } 0 \le k \le n-1 \Bigg\}.$$

(2) Define the norm on $\mathcal{L}_A^2$ by

$$\|X\|_{\mathcal{L}_A^2} = \left(\mathbb{E}\left[\int_0^T |X_s(\omega)|^2 \,\mathrm{d}s\right]\right)^{1/2},$$

then $(\mathcal{L}_A^2, \|\cdot\|_{\mathcal{L}_A^2})$ is a Banach space. $\mathcal{L}_{A,0}^2$ is its linear subspace.

(3) For any $t \in [0,T]$, define

$$L_t^2 = L^2(\Omega, \mathcal{F}_t, \mathbb{P}) = \{X : \Omega \to \mathbb{R} \mid \mathbb{E}(|X_t|^2) < \infty\},$$

the L_t^2 norm is

$$\|X\|_{L_t^2} = [\mathbb{E}(|X|^2)]^{1/2}.$$

$(L_t^2, \|\cdot\|_{L_t^2})$ is also Banach.

(4) For any $0 \le t \le T$, define the operator (Itô integral for simple adapted processes) $I_t : \mathcal{L}_{A,0}^2 \to L_t^2$ by

$$I_t(X)(\omega) = \int_0^t X_s \,\mathrm{d}B_s = \sum_{k=0}^{n-1} c_k(\omega)(B_{t\wedge t_{k+1}}(\omega) - B_{t\wedge t_k}(\omega)), \quad \forall X \in \mathcal{L}_{A,0}^2,$$

in particular,

$$I_T(X)(\omega) = \int_0^T X_s \,\mathrm{d}B_s = \sum_{k=0}^{n-1} c_k(\omega)(B_{t_{k+1}}(\omega) - B_{t_k}(\omega)), \quad \forall X \in \mathcal{L}_{A,0}^2.$$

Proposition 1.2.2 (Properties of Itô integral for simple adapted processes)**.** *Let $I_t : \mathcal{L}_{A,0}^2 \to L_t^2$ be the Itô integral. Then for any $X, Y \in \mathcal{L}_{A,0}^2$,*

(1) *$I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y)$ for all $\alpha, \beta \in \mathbb{R}$.*

(2) *Itô isometry: $\|I_t(X)\|_{L^2} = \|X\|_{\mathcal{L}_{A,0}^2}$.*

(3) *$t \mapsto I_t(X)(\omega)$ is continuous for almost all $\omega \in \Omega$.*

(4) *$\{I_t(X) : t \ge 0\}$ is a martingale and hence $\mathbb{E}(I_t(X)) = \mathbb{E}(I_0(X)) = 0$*

Proof. (1) Clear.

In the following arguments, suppose

$$X_s(\omega) = c_0(\omega)\mathbb{1}_{\{0\}}(s) + \sum_{k=0}^{n-1} c_k(\omega)\mathbb{1}_{(t_k, t_{k+1}]}(s).$$

(2)Let $t = T$,

$$\begin{aligned}
\|I_T(X)\|_{L^2}^2 &= \mathbb{E}\left[\left(\sum_{k=0}^{n-1} c_k(\omega)(B_{t_{k+1}}(\omega) - B_{t_k}(\omega))\right)^2\right] \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathbb{E}\left[c_j c_k (B_{t_{j+1}} - B_{t_j})(B_{t_{k+1}} - B_{t_k})\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\left[c_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}(c_k^2)\mathbb{E}\left[(B_{t_{k+1}} - B_{t_k})^2\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}(c_k^2)(t_{k+1} - t_k) \\
&= \|X\|_{\mathcal{L}^2_{A,0}}^2 .
\end{aligned}$$

in the third " $=$ ", we cancel the non-diagonal terms because if $j < k$,

$$\begin{aligned}
\mathbb{E}\left[c_j c_k (B_{t_{j+1}} - B_{t_j})(B_{t_{k+1}} - B_{t_k})\right] &= \mathbb{E}\left[\mathbb{E}\left[c_j c_k (B_{t_{j+1}} - B_{t_j})(B_{t_{k+1}} - B_{t_k})\middle|\mathcal{F}_k\right]\right] \\
&= \mathbb{E}\left[c_j c_k (B_{t_{j+1}} - B_{t_j})\mathbb{E}\left(B_{t_{k+1}} - B_{t_k}\middle|\mathcal{F}_k\right)\right] \\
&= \mathbb{E}\left[c_j c_k (B_{t_{j+1}} - B_{t_j})\right]\mathbb{E}\left(B_{t_{k+1}} - B_{t_k}\right) = 0.
\end{aligned}$$

(3) Since w.p.1., $t \mapsto B_t(\omega)$ is continuous, then $B_{t\wedge t_k}$ is continuous. The sum of continuous functions is still continuous.

(4) First, by Itô isometry, we have

$$\mathbb{E}[|I_t(X)|^2] = \mathbb{E}\left[\int_0^t |X|^2\,\mathrm{d}s\right] < \infty,$$

so $I_t(X) \in L^2$ and hence in L^1. Second, suppose $0 \le s < t$, we want to show

$$\mathbb{E}[I_t(X)|\mathcal{F}_s] = I_s(X).$$

We can assume $t = T$, then

$$\begin{aligned}\mathbb{E}[I_t(X)|\mathcal{F}_s] &= \mathbb{E}\left[\sum_{k=0}^{n-1} c_k(\omega)(B_{t_{k+1}}(\omega) - B_{t_k}(\omega))\middle|\mathcal{F}_s\right] \\ &= \sum_{k=0}^{n-1} \mathbb{E}[c_k(\omega)(B_{t_{k+1}}(\omega) - B_{t_k}(\omega))|\mathcal{F}_s].\end{aligned}$$

There are three cases:

i) $t_{k+1} \leq s$, then $c_k, B_{t_k}, B_{t_{k+1}} \in \mathcal{F}_s$,

$$\mathbb{E}[c_k(B_{t_{k+1}} - B_{t_k})|\mathcal{F}_s] = c_k(B_{t_{k+1}} - B_{t_k})$$

ii) $t_k \leq s < t_{k+1}$, then $c_k \in \mathcal{F}_{t_k} \subseteq \mathcal{F}_s$,

$$\begin{aligned}\mathbb{E}[c_k(B_{t_{k+1}} - B_{t_k})|\mathcal{F}_s] &= c_k\mathbb{E}[(B_{t_{k+1}} - B_{t_k})|\mathcal{F}_s] \\ &= c_k\mathbb{E}[(B_{t_{k+1}} - B_s)|\mathcal{F}_s] + c_k\mathbb{E}[(B_s - B_{t_k})|\mathcal{F}_s] \\ &= c_k\mathbb{E}[B_{t_{k+1}} - B_s] + c_k(B_s - B_{t_k}) \\ &= c_k(B_s - B_{t_k})\end{aligned}$$

iii) $s < t_k$, then

$$\mathbb{E}[c_k(B_{t_{k+1}} - B_{t_k})|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[c_k(B_{t_{k+1}} - B_{t_k})|\mathcal{F}_{t_k}]|\mathcal{F}_s] = \mathbb{E}[c_k\mathbb{E}[B_{t_{k+1}} - B_{t_k}]|\mathcal{F}_s] = 0.$$

Therefore

$$\mathbb{E}[I_t(X)|\mathcal{F}_s] = \sum_{k=0}^{n-1} \mathbb{E}[c_k(B_{t_{k+1}} - B_{t_k})|\mathcal{F}_s] = \sum_{k=0}^{n-1} c_k(B_{t_{k+1}\wedge s} - B_{t_k\wedge s}) = I_s(X).$$

□

Lemma 1.2.3. *$\mathcal{L}^2_{A,0}$ is dense in $\mathcal{L}^2_A$, i.e. for any $X \in \mathcal{L}^2_A$, there is a sequence $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{L}^2_{A,0}$ s.t.*

$$\lim_{n\to\infty} \|X_n - X\|_{\mathcal{L}^2_A} = \lim_{n\to\infty}\left(\mathbb{E}\left[\int_0^T |X_n(s) - X(s)|^2\,\mathrm{d}s\right]\right)^{1/2} = 0.$$

Theorem 1.2.4 (Itô integral for $X \in \mathcal{L}^2_A$)**.** *For any $X \in \mathcal{L}^2_A$, by Lemma 1.2.3, there is* *sequence* $X_n \in \mathcal{L}^2_{A,0}$ *s.t.*

$$\lim_{n\to\infty} \|X_n - X\|_{\mathcal{L}^2_A} = 0.$$

Then for any fixed $t \in [0,T]$, *there is an* $I_t(X) \in L^2$ *s.t.*

$$\lim_{n\to\infty} \|I_t(X) - I_t(X_n)\|_{L^2} = 0.$$

Moreover, $I_t(X)$ *is unique, i.e. independent of the choice of* X_n. *We call* $I_t(X)$ *the Itô integral for* $X \in \mathcal{L}^2_A$, *denoted as*

$$I_t(X)(\omega) = \int_0^t X_s(\omega)\,\mathrm{d}B_s(\omega).$$

Proof. Since $X_n \to X$ in $\mathcal{L}^2_A$, it is a Cauchy sequence, i.e. for any $\varepsilon > 0$, there is $N_1 > 0$, s.t. for all $m, n \geq N_1$,

$$\|X_m - X_n\|_{\mathcal{L}^2_A} < \varepsilon.$$

By Itô isometry for simple adapted processes,

$$\|I_t(X_m) - I_t(X_n)\|_{L^2} = \|I_t(X_m - X_n)\|_{L^2} = \|X_m - X_n\|_{\mathcal{L}^2_A} < \varepsilon,$$

i.e. $\{I_t(X_n)\}_{n=1}^\infty$ is a Cauchy sequence in L^2. Since L^2 is complete, there is $I_t(X) \in L^2$ s.t. $I_t(X_n) \to I_t(X)$ in L^2.

Uniqueness. Suppose another sequence of simple adapted processes $X'_n \to X$ in $\mathcal{L}^2_A$. Let $I'_t(X) \in L^2$ s.t.

$$\left\|I_t(X'_n) - I_t(X)\right\|_{L^2} \to 0.$$

Then

$$\left\|X_n - X'_n\right\|_{\mathcal{L}^2_A} \leq \|X_n - X\|_{\mathcal{L}^2_A} + \left\|X - X'_n\right\|_{\mathcal{L}^2_A} \to 0,$$

by Itô isometry,

$$\left\|I_t(X_n) - I_t(X'_n)\right\|_{L^2} = \left\|I_t(X_n - X'_n)\right\|_{L^2} = \left\|X_n - X'_n\right\|_{\mathcal{L}^2_A} \to 0.$$

Therefore

$$\left\|I_t(X) - I'_t(X)\right\|_{L^2} \leq \|I_t(X) - I_t(X_n)\|_{L^2} + \left\|I_t(X_n) - I_t(X'_n)\right\|_{L^2} + \left\|I_t(X'_n) - I'(X)\right\|_{L^2} \to 0,$$

i.e. $I_t(X) = I'_t(X)$ almost surely (for almost all $\omega \in \Omega$). $\square$

Remark. 1. For each $t \in [0,T]$, the sequence $I_t(X_n)$ converges in L^2 to a limit I_t, which is unique up to almost sure equivalence. The family $\{I_t(\omega) : t \in [0,T]\}$ defines a stochastic process.
2. This process may admit different modifications (or called version). That is, if $\{I'_t(\omega) : t \in [0,T]\}$ is another process such that for each t,

$$\mathbb{P}(I'_t \neq I_t) = 0,$$

then $\{I'_t\}$ is a modification of $\{I_t\}$.
3. Let L^2_m denote the space of equivalence classes of stochastic processes under modification (i.e., processes that agree almost surely at each time). Then the Itô integral defines an operator

$$I : \mathcal{L}^2_A \to L^2_m, \quad X \mapsto I(X)(t,\omega), \quad \forall X \in \mathcal{L}^2_A.$$

Proposition 1.2.5 (Properties of Itô integral for adapted processes)**.** *Let $I : \mathcal{L}^2_A \to L^2_m$ be the Itô integral. Then for any $X, Y \in \mathcal{L}^2_A$,*

(1) $I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y)$ *for all* $\alpha, \beta \in \mathbb{R}$.

(2) *Itô isometry:* $\|I_t(X)\|_{L^2} = \|X\|_{\mathcal{L}^2_{A,0}}$.

(3) *There is a modification* $\tilde{I}_t(X)$ *of* $I_t(X)$ *s.t.* $t \mapsto \tilde{I}_t(X)(\omega)$ *is continuous for almost all* $\omega \in \Omega$.

(4) $\{I_t(X) : t \geq 0\}$ *is a martingale and hence* $\mathbb{E}(I_t(X)) = \mathbb{E}(I_0(X)) = 0$

Proof. (1)It's clear by approximation. (2) Suppose $X_n \in \mathcal{L}^2_{A,0}$ converges to X in $\mathcal{L}^2_A$-norm, then by Theorem 1.2.4 and Itô isometry for simple adapted processes,

$$\|I_t(X)\|_{L^2} = \lim_{n\to\infty} \|I_t(X_n)\|_{L^2} = \lim_{n\to\infty} \|X_n\|_{\mathcal{L}^2_A} = \|X\|_{\mathcal{L}^2_A}.$$

(3) Step 0. For any X_n, $t \mapsto I_t(X_n)(\omega)$ is continuous a.s., we will show that for each path ω, there is a subsequence $I_t(X_{n_j})(\omega) \to I_t(X)(\omega)$ uniformly, then by Weierstrass uniform convergence theorem, $t \mapsto I_t(X)(\omega)$ is continuous. Therefore we want to show $\{I_t(X_n)\}_{n=1}^{\infty}$

is Cauchy w.r.t. the norm

$$\|f\|_\infty = \sup_t |f(t)|.$$

Step 1. For any $m, n \in \mathbb{Z}_+$, since $I_t(X_m) - I_t(X_n) = I_t(X_m - X_n)$ is a martingale w.r.t. $\mathcal{F}_t$, $|I_t(X_m) - I_t(X_n)|^2$ is a submartingale, then by Doob's maximal inequality, for any $k \in \mathbb{Z}_+$,

$$\mathbb{P}\left(\sup_{0\le t\le T} |I_t(X_m) - I_t(X_n)| \ge \frac{1}{2^k}\right) \le 2^{2k}\mathbb{E}\left[|I_t(X_m) - I_t(X_n)|^2\right] = 2^{2k}\|X_m - X_n\|^2_{\mathcal{L}^2_{A,0}}.$$

Step 2. Since we assume $X_n \to X$ in $\mathcal{L}^2_A$, $\{X_n\}_{n=1}^\infty$ is Cauchy w.r.t. the norm $\|\cdot\|_{\mathcal{L}^2_{A,0}}$, then by definition, for any k, we can find $N_k \in \mathbb{Z}_+$ s.t. for all $m, n \ge N_k$,

$$\|X_m - X_n\|^2_{\mathcal{L}^2_{A,0}} \le \frac{1}{2^{3k}},$$

then

$$\mathbb{P}\left(\sup_{0\le t\le T} |I_t(X_m) - I_t(X_n)| \ge \frac{1}{2^k}\right) \le \frac{2^{2k}}{2^{3k}} = \frac{1}{2^k}.$$

In particular,

$$\mathbb{P}\left(\sup_{0\le t\le T} |I_t(X_{N_k}) - I_t(X_{N_k+1})| \ge \frac{1}{2^k}\right) \le \frac{1}{2^k}.$$

By Borel-Cantelli lemma, since

$$\sum_{k=1}^\infty \mathbb{P}\left(\sup_{0\le t\le T} |I_t(X_{N_k}) - I_t(X_{N_k+1})| \ge \frac{1}{2^k}\right) \le \sum_{k=1}^\infty \frac{1}{2^k} = 1 < \infty,$$

we have

$$\mathbb{P}\left(\sup_{0\le t\le T} |I_t(X_{N_k}) - I_t(X_{N_k+1})| \ge \frac{1}{2^k} \quad i.o.\right) = 0,$$

i.e. there is $k_0 \in \mathbb{Z}_+$, for any $k \ge k_0$,

$$\mathbb{P}\left(\sup_{0\le t\le T} |I_t(X_{N_k}) - I_t(X_{N_k+1})| \le \frac{1}{2^k}\right) = 1,$$

i.e.

$$\|I_t(X_{N_k}) - I_t(X_{N_k+1})\|_\infty \le \frac{1}{2^k}, \quad a.s.$$

Step 3. For any $\varepsilon > 0$, let k_1 be large enough s.t.

$$\frac{1}{2^{k_1-1}} < \varepsilon,$$

let $K = \max\{k_0, k_1\}$, then for any $j > k \geq K$,

$$\begin{aligned}\left\|I_t(X_{N_j}) - I_t(X_{N_k})\right\|_\infty &\leq \left\|I_t(X_{N_j}) - I_t(X_{N_j - 1})\right\|_\infty + \cdots + \|I_t(X_{N_k+1}) - I_t(X_{N_k})\|_\infty \\ &\leq \frac{1}{2^{j-1}} + \cdots + \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &\leq \frac{1}{2^k}\left(\sum_{l=0}^{\infty} \frac{1}{2^l}\right) \\ &\leq \frac{1}{2^{k-1}} < \varepsilon,\end{aligned}$$

therefore the subsequence $\{I_t(X_{N_k})\}_{k=1}^{\infty}$ is Cauchy a.s., hence its limit

$$\lim_{k\to\infty} I_t(X_{N_k})$$

is continuous a.s.

Step 4. Show $\lim_{k\to\infty} I_t(X_{N_k})$ is a continuous version of $I_t(X)$.

Since for each t, $I_t(X_n) \to I_t(X)$ in L^2, hence also in probability, any a.s. convergent subsequence must have the same limit $I_t(X)$, therefore

$$\lim_{k\to\infty} I_t(X_{N_k}) = I_t(X) \quad \forall t \in [0, T] \quad a.s.$$

Then

$$t \mapsto \lim_{k\to\infty} I_t(X_{N_k})$$

is a continuous version of $\{I_t(X) : t \in [0, T]\}$. □

Corollary 1.2.6 (Quadratic variation)**.** *Let $X \in \mathcal{L}^2_A$, then the quadratic variation of $I_t(X)$ is*

$$[X](t) := \lim_{|\Gamma|\to 0} V^2(I_t(X), [0, t], \Gamma) = \int_0^t X_s^2(\omega)\,\mathrm{d}s, \quad \textit{in probability.}$$

Proof. Suppose $\Gamma = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, then

$$V^2(I_t(X), [0, t], \Gamma) = \sum_{k=0}^{n-1} |I_{t_{k+1}}(X) - I_{t_k}(X)|^2.$$

By Itô isometry, we have

$$\mathbb{E}\left(|I_{t_{k+1}}(X) - I_{t_k}(X)|^2\right) = \mathbb{E}\left(\left|\int_{t_k}^{t_{k+1}} X_s \,\mathrm{d}B_s\right|^2\right) = \mathbb{E}\left(\int_{t_k}^{t_{k+1}} X_s^2 \,\mathrm{d}s\right),$$

so

$$\mathbb{E}[V^2(I_t(X), [0,t], \Gamma)] = \mathbb{E}\left(\int_0^t X_s^2 \,\mathrm{d}s\right),$$

therefore as $|\Gamma| \to 0$,

$$V^2(I_t(X), [0,t], \Gamma) \to \int_0^t X_s^2 \,\mathrm{d}s, \quad \text{in } L^1,$$

hence also in probability. □

Corollary 1.2.7. *For any $X \in \mathcal{L}_A^2$,*

$$I_t^2(X) - [X](t) = I_t^2(X) - \int_0^t X_s^2 \,\mathrm{d}s$$

is a martingale, i.e. for any $0 \le s < t$,

$$\mathbb{E}\left[I_t^2(X) - [X](t) \middle| \mathcal{F}_s\right] = I_s^2(X) - [X](s).$$

Proof. Direct from the definition of quadratic variation (1.1.5). □

1.3 Itô formula

Theorem 1.3.1 (Itô formula)**.** *If $f : \mathbb{R} \to \mathbb{R} \in C^2$, then w.p.1.,*

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s) \,\mathrm{d}s.$$

Proof. Fix partition $\Gamma = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, then

$$f(B_t) - f(B_0) = \sum_{k=0}^{n-1} [f(B_{t_{k+1}} - f(B_{t_k}))].$$

Claim. As $|\Gamma| \to 0$,

$$\sum_{k=0}^{n-1} [f(B_{t_{k+1}} - f(B_{t_k}))] \to \int_0^t f'(B_s) \,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s) \,\mathrm{d}s$$

in probability.

Remark: this claim implies Itô formula. If we know $X = X_n \to X_\infty$ in probability as $n \to \infty$, then for any $\varepsilon > 0$,

$$\mathbb{P}(|X - X_\infty| > \varepsilon) = \lim_{n\to\infty} \mathbb{P}(|X_n - X_\infty| > \varepsilon) = 0,$$

thus

$$\mathbb{P}(X = X_\infty) = \mathbb{P}(\bigcap_{m=1}^{\infty} |X - X_\infty| \le \frac{1}{m}) = \lim_{m\to\infty} \mathbb{P}(|X - X_\infty| \le \frac{1}{m}) = 1.$$

Proof of the Claim. We will prove a weak version, i.e. suppose

$$\sup_{x\in\mathbb{R}}(|f(x)| + |f'(x)|) < \infty.$$

By Taylor's theorem,

$$\sum_{k=0}^{n-1}[f(B_{t_{k+1}} - f(B_{t_k}))] = \sum_{k=0}^{n-1} f'(B_{t_k})(B_{t_{k+1}} - B_{t_k}) + \frac{1}{2}\sum_{k=0}^{n-1} f''(z_k)(B_{t_{k+1}} - B_{t_k})^2 := S_1 + \frac{S_2}{2},$$

for some $z_k(\omega)$ between $B_{t_k}(\omega)$ and $B_{t_{k+1}}(\omega)$.

We will show as $|\Gamma| \to 0$,

$$S_1 \to \int_0^t f'(B_s)\,\mathrm{d}B_s = I_t(f'(B)), \quad S_2 \to \int_0^t f''(B_s)\,\mathrm{d}s, \quad \text{in probability.}$$

(1) For S_1, we will show $S_1 \to I_t(f'(B))$ in L^2 hence in probability. Let

$$X_s(\omega) = \sum_{k=0}^{n-1} f'(B_{t_k}(\omega))\mathbb{1}_{(t_k,t_{k+1}]}(s),$$

then $X \in \mathcal{L}^2_{A,0}$ and $S_1 = I_t(X)$. By Itô isometry,

$$\begin{aligned}
\mathbb{E}[|S_1 - I_t(f'(B))|^2] &= \mathbb{E}[|I_t(X - f'(B))|^2] = \int_0^t \mathbb{E}[|X_s - f'(B_s)|^2]\,\mathrm{d}s \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[|f'(B_{t_k}) - f'(B_s)|^2]\,\mathrm{d}s \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[|f'(c)|^2 |B_{t_k} - B_s|^2]\,\mathrm{d}s \qquad \text{(by the mean value theorem)} \\
&\le M^2 \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (s - t_k)\,\mathrm{d}s \\
&= \frac{M^2}{2} \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \\
&\le M^2 t |\Gamma| \\
&\to 0, \qquad \text{as } |\Gamma| \to 0
\end{aligned}$$

(2)For S_2, we have

$$S_2 = \sum_{k=0}^{n-1} f''(B_{t_k})(B_{t_{k+1}} - B_{t_k})^2 + \sum_{k=0}^{n-1} [f''(z_k) - f''(B_{t_k})](B_{t_{k+1}} - B_{t_k})^2 := S_3 + S_4,$$

For S_3,

$$S_3 = \sum_{k=0}^{n-1} f''(B_{t_k})(t_{k+1} - t_k) + \sum_{k=0}^{n-1} f''(B_{t_k})[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)] =: S_5 + S_6.$$

The Riemann sum

$$\lim_{|\Gamma| \to 0} S_5 = \int_0^t f''(B_s)\,\mathrm{d}s, \quad a.s.$$

and $S_6 \to 0$ in L^2 by computing $\mathbb{E}[S_6^2]$.

$$\begin{aligned}
\mathbb{E}[S_6^2] &= \mathbb{E}\left[\left(\sum_{k=0}^{n-1} f''(B_{t_k})[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\right)^2\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)][(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\right] \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathbb{E}\left[f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)][(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\right]
\end{aligned}$$

For the case $j \neq k$, e.g. $j < k$, we have $t_j < t_k$, so conditioning on $\mathcal{F}_{t_k}$,

$$\begin{aligned}
&\mathbb{E}\left[f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)][(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)][(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\middle|\mathcal{F}_{t_k}\right]\right] \\
&= \mathbb{E}\left[f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)]\mathbb{E}\left[[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\middle|\mathcal{F}_{t_k}\right]\right] \\
&= \mathbb{E}\left[f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)]\mathbb{E}\left[[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\right]\right] \\
&= \mathbb{E}\left[f''(B_{t_j})f''(B_{t_k})[(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)]\right]\mathbb{E}\left[[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]\right] \\
&= 0,
\end{aligned}$$

so only the term $j = k$ remains in the sum, i.e.

$$\begin{aligned}
\mathbb{E}[S_6^2] &= \sum_{k=0}^{n-1} \mathbb{E}\left[|f''(B_{t_k})|^2[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]^2\right] \\
&\leq C^2 \sum_{k=0}^{n-1} \mathbb{E}\left[[(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)]^2\right] \qquad \text{(assume } |f''| \leq C\text{)} \\
&= C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \mathbb{E}\left[\left(\frac{(B_{t_{k+1}} - B_{t_k})^2}{t_{k+1} - t_k} - 1\right)^2\right] \\
&= C^2 \mathbb{E}[|\xi^2 - 1|^2] \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \qquad (\xi \sim \mathcal{N}(0,1)) \\
&= C^2 \mathbb{E}[|\xi^2 - 1|^2] \cdot |\Gamma| t \to 0 \qquad \text{as } |\Gamma| \to 0.
\end{aligned}$$

For S_4, since B_t is continuous w.p.1., there is $r_k \in [t_k, t_{k+1}]$ s.t. B_{r_k} is between B_{t_k} and $B_{t_{k+1}}$, thus

$$|S_4| \leq \sum_{k=0}^{n-1} |f''(B_{r_k}) - f''(B_{t_k})|(B_{t_{k+1}} - B_{t_k})^2 \leq V^2(B, \Gamma, [0,t]) \cdot \max_k |f''(B_{r_k}) - f''(B_{t_k})|.$$

By the uniform continuity of $s \mapsto f''(B_s)$ on $[0,t]$,

$$\lim_{|\Gamma| \to 0} \max_k |f''(B_{r_k}) - f''(B_{t_k})| = 0, \quad a.s.$$

and by the quadratic variation of BM,

$$V^2(B, \Gamma, [0,t]) \to t \quad \text{in } L^2,$$

Since convergence a.s. and in L^2 both imply convergence in probability, and convergence in probability is linear, so $S_4 \to 0$ in probability. □

Remark. 1. For simplicity, we sometimes denote the formula as

$$\mathrm{d}f(B_t) = f'(B_t)\,\mathrm{d}B_t + \frac{1}{2} f''(B_t)\,\mathrm{d}t.$$

Example 1.3.2. By Itô formula, let $f = x^2/2$ we have

$$\frac{B_t^2}{2} - 0 = \int_0^t B_s\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{d}s,$$

therefore

$$\int_0^t B_s\,\mathrm{d}B_s = \frac{B_t^2}{2} - \frac{t}{2}.$$

Theorem 1.3.3. *Suppose $f(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R} \in C^{1,2}$, then w.p.1.*

$$f(t, B_t) - f(0, B_0) = \int_0^t \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right](s, B_s)\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\,\mathrm{d}B_s.$$

Corollary 1.3.4. *If $f(t,x)$ is a polynomial in t, x with*

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0,$$

then $f(t, B_t)$ is a martingale and $\mathbb{E}[f(t, B_t)] = f(0, B_0)$.

Proof. By Theorem 1.3.3,

$$f(t, B_t) - f(0, B_0) = \int_0^t \frac{\partial f}{\partial x}(s, B_s)\,\mathrm{d}B_s.$$

We want to show

$$\mathbb{E}\left[\int_0^T \left|\frac{\partial f}{\partial x}(s, B_s)\right|^2 \mathrm{d}s\right] = \int_0^T \mathbb{E}\left[\left|\frac{\partial f}{\partial x}(s, B_s)\right|^2\right] \mathrm{d}s < \infty,$$

Since $f(t, x)$ is a polynomial, we can write

$$\left|\frac{\partial f}{\partial x}(s, B_s)\right|^2 = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} s^i B_s^j \leq C(1 + s^m)(1 + B_s^n),$$

and

$$\mathbb{E}\left[\left|\frac{\partial f}{\partial x}(s, B_s)\right|^2\right] \leq \mathbb{E}[C(1 + s^m)(1 + B_s^n)] = C(1 + s^m)(1 + \mathbb{E}(B_s^n)) \leq C(1 + s^m)(1 + C_1 s^{n/2}),$$

then its integral is finite. Therefore

$$\int_0^t \frac{\partial f}{\partial x}(s, B_s)\,\mathrm{d}B_s$$

is an Itô integral, hence a martingale. □

Example 1.3.5. Let $\alpha \in \mathbb{R}$, define

$$X_t = X_0 \exp(\alpha B_t - \frac{1}{2}\alpha^2 t).$$

Let $f(t, x) = X_0 \exp(\alpha x - \frac{1}{2}\alpha^2 t)$

$$\begin{aligned}
\mathrm{d}X_s = \mathrm{d}f(B_s, s) &= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right](s, B_s)\,\mathrm{d}s + \frac{\partial f}{\partial x}(s, B_s)\,\mathrm{d}B_s \\
&= (-\frac{\alpha^2}{2}X_s + \frac{1}{2}\alpha^2 X_s)\,\mathrm{d}s + \alpha X_s\,\mathrm{d}B_s \\
&= \alpha X_s\,\mathrm{d}B_s.
\end{aligned}$$

Therefore X_t defined above satisfies the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \alpha X_t\,\mathrm{d}B_t, \quad X(0) = X_0.$$

1.4 Itô formula for Itô processes

Definition 1.4.1. Let $\mathcal{F}_t$ be the filtration s.t. B_t adapted to it. Suppose $\mu(s,\omega)$ and $\sigma(s,\omega)$ are adapted processes w.r.t. $\mathcal{F}_t$ and satisfy the usual condition:

$$\mathbb{P}\left(\int_0^t |\mu|\,\mathrm{d}s < \infty\right) = 1, \quad \mathbb{P}\left(\int_0^t |\sigma|^2\,\mathrm{d}s < \infty\right) = 1.$$

We call $Z(t,\omega)$ (or $Z_t(\omega)$) an Itô process if it is defined by

$$Z(t,\omega) = Z(0,\omega) + \int_0^t \mu(s,\omega)\,\mathrm{d}s + \int_0^t \sigma(s,\omega)\,\mathrm{d}B_s,$$

and we denote it as

$$\mathrm{d}Z = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B.$$

μ_s is called the drift term and σ_s is the diffusion coefficient.

Remark. The quadratic variation for Z_t is

$$[Z,Z](t) = \int_0^t \sigma_s^2\,\mathrm{d}s.$$

Theorem 1.4.2. *Suppose $f \in C^2$ and Z_t is an Itô process, then w.p.1.*

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)\mu_s\,\mathrm{d}s + \int_0^t f'(Z_s)\sigma_s\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(Z_s)\sigma_s^2\,\mathrm{d}s.$$

Theorem 1.4.3. *Suppose $f(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R} \in C^{1,2}$ and Z_t is an Itô process. Then w.p.1.*

$$f(t,Z_t) - f(0,Z_0) = \int_0^t \left[\frac{\partial f}{\partial t}(s,Z_s) + \mu_s\frac{\partial f}{\partial x}(s,Z_s) + \frac{1}{2}\sigma_s^2\frac{\partial^2 f}{\partial x^2}(s,Z_s)\right]\mathrm{d}s + \int_0^t \sigma_s\frac{\partial f}{\partial x}(s,Z_s)\,\mathrm{d}B_s.$$

Example 1.4.4 (Ornstein-Unlenbeck process)**.** Let $\alpha,\sigma > 0$, define the Ornstein-Unlenbeck process by

$$X_t = X_0 e^{-\alpha t} + \sigma\int_0^t e^{-\alpha(t-s)}\,\mathrm{d}B_s.$$

Let

$$Z_t = \int_0^t e^{\alpha s}\,\mathrm{d}B_s,$$

and $f(t,x) = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} x$, then $X_t = f(t, Z_t)$. By the Itô formula,

$$\begin{aligned} \mathrm{d}X_s &= \left[\frac{\partial f}{\partial t}(s, Z_s) + \frac{1}{2} e^{2\alpha s} \frac{\partial^2 f}{\partial x^2}(s, Z_s)\right] \mathrm{d}s + e^{\alpha s} \frac{\partial f}{\partial x}(s, Z_s)\,\mathrm{d}B_s \\ &= \frac{\partial f}{\partial t}(s, Z_s)\,\mathrm{d}s + e^{\alpha s} \sigma e^{-\alpha s}\,\mathrm{d}B_s \\ &= -\alpha X_0 e^{-\alpha s} + -\alpha\sigma e^{-\alpha s} Z_s + \sigma\,\mathrm{d}B_s \\ &= -\alpha X_s\,\mathrm{d}s + \sigma\,\mathrm{d}B_s. \end{aligned}$$

Therefore, X_t defined above satisfies the SDE

$$\mathrm{d}X_t = -\alpha X_t\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \qquad X(0) = X_0.$$

1.5 Multi-dimensional Itô formula

Definition 1.5.1. $\boldsymbol{B}(t) = (B^{(1)}(t), B^{(2)}(t), \cdots, B^{(d)}(t))$ is called a d-dimensional BM if $\{B^{(i)}(t)\}_{i=1}^d$ are independent 1-d BM. Define the Brownian filtration by

$$\mathcal{F}_t^B = \sigma(B^{(i)}(s), 1 \le i \le d, 0 \le s \le t).$$

Theorem 1.5.2. *For d-dimensional BM $\boldsymbol{B} = (B^{(1)}, \cdots, B^{(d)})$, let $f \in C^2(\mathbb{R}^d; \mathbb{R})$, then*

$$f(\boldsymbol{B}_t) - f(\boldsymbol{B}_0) = \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial z_i}(\boldsymbol{B}_s)\,\mathrm{d}B_s^{(i)} + \frac{1}{2} \sum_{i=1}^d \int_0^t \frac{\partial^2 f}{\partial z_i^2}(\boldsymbol{B}_s)\,\mathrm{d}s.$$

Example 1.5.3. Let $d > 2$, for d-dimensional BM $\boldsymbol{B} = (B^{(1)}, \cdots, B^{(d)})$, define

$$|\boldsymbol{B}_t| = \left(\sum_{k=1}^d (B_t^{(k)})^2\right)^{1/2}.$$

Let $f(z_1, \cdots, z_d) = (z_1^2 + \cdots + z_d^2)^{1/2}$, then $|\boldsymbol{B}_t| = f(\boldsymbol{B}_t)$. Since,

$$\frac{\partial f}{\partial z_i} = \frac{z_i}{f}, \quad \frac{\partial^2 f}{\partial z_i^2} = \frac{f - \frac{z_i^2}{f}}{f^2} = \frac{f^2 - z_i^2}{f^3},$$

by Itô's formula, we have

$$\begin{aligned} f(\boldsymbol{B}_t) - f(\boldsymbol{B}_0) &= \sum_{i=1}^{d} \int_0^t \frac{B_s^{(i)}}{f(\boldsymbol{B}_s)} \, \mathrm{d}B_s^{(i)} + \frac{1}{2} \sum_{i=1}^{d} \int_0^t \frac{f^2(\boldsymbol{B}_s) - (B_s^{(i)})^2}{f^3(\boldsymbol{B}_s)} \, \mathrm{d}s \\ &= \sum_{i=1}^{d} \int_0^t \frac{B_s^{(i)}}{f(\boldsymbol{B}_s)} \, \mathrm{d}B_s^{(i)} + \frac{1}{2} \int_0^t \frac{df^2(\boldsymbol{B}_s) - \sum_{i=1}^{d} (B_s^{(i)})^2}{f^3(\boldsymbol{B}_s)} \, \mathrm{d}s \\ &= \sum_{i=1}^{d} \int_0^t \frac{B_s^{(i)}}{f(\boldsymbol{B}_s)} \, \mathrm{d}B_s^{(i)} + \frac{1}{2} \int_0^t \frac{d-1}{f(\boldsymbol{B}_s)} \, \mathrm{d}s, \end{aligned}$$

i.e. $|\boldsymbol{B}_s|$ is the solution to the SDE

$$\mathrm{d}X_t = \sum_{i=1}^{d} \frac{B_t^{(i)}}{X_t} \, \mathrm{d}B_t^{(i)} + \frac{d-1}{2X_t} \, \mathrm{d}t.$$

Chapter 2

Applications

2.1 Exit time and exit distribution for diffusion processes

2.1.1 1-dimension

Let (l, r) be an open real interval, consider the following 1-dimensional diffusion process:

$$\begin{cases} \mathrm{d}X_t = v(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t \\ X_0 = x \in (l, r). \end{cases}$$

Define the operator

$$\mathcal{L}f(x) = \frac{\sigma^2(x)}{2} f''(x) + v(x)f(x).$$

Theorem 2.1.1. *Let $[a, b] \subseteq (l, r)$, suppose the diffusion starts at $X_0 = x \in [a, b]$. Let*

$$\tau = \inf\{t \geq 0 : X_t \notin [a, b]\}.$$

Then the unique solution $p(x)$ to the ODE

$$\begin{cases} \mathcal{L}p(x) = -1 \\ p(a) = p(b) = 0 \end{cases}$$

satisfies

$$p(x) = \mathbb{E}_x(\tau)$$

Proof. Apply Itô's formula to $p(X_t)$ on $[0,\tau]$, then we have

$$\begin{aligned} p(X_{\tau\wedge t}) &= p(X_0) + \int_0^{\tau\wedge t} p'(X_s)\sigma(X_s)\,\mathrm{d}B_s + \int_0^{\tau\wedge t} \mathcal{L}p(X_s)\,\mathrm{d}s \\ &= p(x) + \int_0^{\tau\wedge t} p'(X_s)\sigma(X_s)\,\mathrm{d}B_s - \tau\wedge t, \end{aligned}$$

Let $t\to\infty$, we have

$$\mathbb{E}(\tau) = p(x).$$

□

Define $u:(l,r)\to\mathbb{R}$ by

$$u(x) = \int_{x_0}^{x} \exp\left(-2\int_{x_0}^{y} v(r)/\sigma^2(r)\,\mathrm{d}r\right)\mathrm{d}y,$$

where $x_0\in(l,r)$ is an arbitrary point, then u is the solution to the ODE:

$$\begin{cases} \mathcal{L}u(x) = 0, \\ u(x_0) = 0 \\ u'(x_0) > 0. \end{cases}$$

For any $[a,b]\subset(l,r)$, define

$$h(x) = \frac{u(x)-u(a)}{u(b)-u(a)},$$

then h is the solution to the ODE for all $x\in[a,b]$,

$$\begin{cases} \mathcal{L}h(x) = 0, \\ h(a) = 0 \\ h(b) = 1. \end{cases}$$

Define

$$\tau_a = \inf\{t\geq 0 : X_t = a\},\quad \tau_b = \inf\{t\geq 0 : X_t = b\}.$$

Then we have the following result.

Theorem 2.1.2. $\mathbb{P}_x(X_\tau = a) = \mathbb{P}_x(\tau_a < \tau_b) = 1 - h(x)$ *and* $\mathbb{P}_x(X_\tau = b) = \mathbb{P}_x(\tau_b < \tau_a) = h(x)$.

Proof. Apply Itô's formula to $u(X_t)$ on $[0,\tau]$, then we have

$$\begin{aligned} u(X_{\tau\wedge t}) &= u(X_0) + \int_0^{\tau\wedge t} u'(X_s)\sigma(X_s)\,\mathrm{d}B_s + \int_0^{\tau\wedge t} \mathcal{L}u(X_s)\,\mathrm{d}s \\ &= u(x) + \int_0^{\tau\wedge t} u'(X_s)\sigma(X_s)\,\mathrm{d}B_s, \end{aligned}$$

then

$$\mathbb{E}[u(X_{\tau\wedge t})] = u(x).$$

Let $t\to\infty$, we have

$$u(x) = \mathbb{E}[u(X_\tau)] = u(a)\mathbb{P}_x(X_\tau = a) + b\mathbb{P}_x(X_\tau = b).$$

Since $1 = \mathbb{P}_x(X_\tau = a) + \mathbb{P}_x(X_\tau = b)$, we have

$$\mathbb{P}_x(X_\tau = a) = \frac{u(b)-u(x)}{u(b)-u(a)}, \quad \mathbb{P}_x(X_\tau = b) = \frac{u(x)-u(a)}{u(b)-u(a)}.$$

□

Example 2.1.3. Let $D=[0,R]$ for some $R>0$. Consider the diffusion process

$$\begin{cases} \mathrm{d}X_t = v\,\mathrm{d}t + \mathrm{d}B_t \\ X_0 = x \in [0,R], \end{cases}$$

where $v\in\mathbb{R}\setminus\{0\}$ is a constant.

(1) For the exit time, solving the ODE

$$\begin{cases} \mathcal{L}p(x) = \frac{1}{2}p''(x) + vp(x) = -1 \\ p(0) = p(R) = 0, \end{cases}$$

we have

$$\mathbb{E}_x(\tau_D) = p(x) = \frac{-R}{v(1-e^{-2vR})}\left(e^{-2vx}-1\right) - \frac{1}{v}x$$

(2) For the exit distribution, we have

$$u(x) = \int_0^x \exp\left(-2\int_0^y v\,\mathrm{d}r\right)\,\mathrm{d}y = \int_0^x e^{-2vy}\,\mathrm{d}y = -\frac{e^{-2vx}}{2v} + \frac{1}{2v},$$

then we have

$$h(x) = \frac{1 - e^{-2vx}}{1 - e^{-2vR}},$$

therefore

$$\mathbb{P}_x(\tau_0 < \tau_R) = 1 - h(x) = \frac{e^{-2vx} - e^{-2vR}}{1 - e^{-2vR}},$$

let $R \to \infty$, we have

$$\mathbb{P}_x(\tau_0 < \infty) = e^{-2vx}.$$

In other words, as the starting point increases, the probability that X_t reaches 0 decreases exponentially.

2.1.2 d-dimension

Consider the d-dimensional diffusion process X_t.

Theorem 2.1.4. *Let $D \subseteq \mathbb{R}^d$ be an open set, suppose X_t starts at $X_0 = x \in D$, define*

$$\tau = \inf\{t \geq 0 : X_t \notin D\}.$$

Then the unique solution $p(x)$ to the PDE

$$\begin{cases} \mathcal{L}p(x) = -1, & x \in D \\ p(x) = 0, & x \in \partial D \end{cases}$$

satisfies

$$p(x) = \mathbb{E}_x(\tau).$$

We can also calculate the exit distribution for some special cases, see the following example.

Example 2.1.5. Let B_t be a standard d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$, and let $0 < r < R$. Define the open annulus domain

$$D := \{x \in \mathbb{R}^d : r < |x| < R\}.$$

Let $\tau = \inf\{t \geq 0 : B_t \notin D\}$ be the first exit time from D. Consider the boundary value problem:

$$\begin{cases} \mathcal{L}u(x) = \frac{1}{2}\Delta u(x) = 0, & x \in D, \\ u(x) = 0, & |x| = r, \\ u(x) = 1, & |x| = R. \end{cases}$$

Then the function $u(x)$ gives the probability that Brownian motion starting at x exits the domain D through the outer boundary $|x| = R$, i.e.,

$$u(x) = \mathbb{P}_x(|B_\tau| = R).$$

Consequently, the complementary probability is:

$$\mathbb{P}_x(|B_\tau| = r) = 1 - u(x).$$

Since the problem is radially symmetric, $u(x)$ depends only on $\rho = |x|$. Let $u(x) = v(\rho)$. The PDE reduces to the ODE:

$$\frac{1}{2}\left(v''(\rho) + \frac{d-1}{\rho}v'(\rho)\right) = 0, \quad r < \rho < R.$$

Solving this, we get:

- For $d = 2$:
$$v(\rho) = \frac{\log(\rho/r)}{\log(R/r)}.$$
- For $d \geq 3$:
$$v(\rho) = \frac{\rho^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}}.$$

Therefore,

$$\mathbb{P}_x(|B_\tau| = r) = 1 - u(x) = \begin{cases} 1 - \dfrac{\log(|x|/r)}{\log(R/r)}, & d = 2, \\ 1 - \dfrac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}}, & d \geq 3. \end{cases}$$

2.2 Lévy's characterization of BM

Theorem 2.2.1 (Lévy). *Let X_t with $X_0 = 0$ be a continuous martingale w.r.t. $\mathcal{F}_t$. If $[X]_t = t$, then X_t is a standard BM.*

Proof. Step 1. We only need to show for any $0 \le s < t$:

(1) $X_t - X_s$ is independent of $\mathcal{F}_s$;

(2) $X_t - X_s \sim \mathcal{N}(0, t-s)$.

Claim: The above two statements hold if and only if for any $u \in \mathbb{R}$,

$$\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_s] = e^{-u^2(t-s)/2}. \tag{2.1}$$

The "only if" part is obvious. We will show the "if" part. Suppose (2.1) holds, first

$$\varphi_{X_t - X_s}(u) = \mathbb{E}[e^{iu(X_t - X_s)}] = \mathbb{E}[\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_s]] = \mathbb{E}[e^{-u^2(t-s)/2}] = e^{-u^2(t-s)/2},$$

by the property of the characteristic function, $X_t - X_s \sim \mathcal{N}(0, t-s)$.
Second, by the definition of conditional expectation, for any $A \in \mathcal{F}_s$,

$$\mathbb{E}[e^{iu(X_t - X_s)}\mathbb{1}_A] = \mathbb{E}[e^{-u^2(t-s)/2}\mathbb{1}_A] = e^{-u^2(t-s)/2}\mathbb{P}(A).$$

Then for any bounded r.v. $Y \in \mathcal{F}_s$, it can be approximated by simple r.v. $Y_n \in \mathcal{F}_s$, thus

$$\mathbb{E}[e^{iu(X_t - X_s)}Y] = e^{-u^2(t-s)/2}\mathbb{E}(Y),$$

For any $B \in \mathcal{F}_s$, let $Y = e^{iw\mathbb{1}_B}$, we have

$$\mathbb{E}[e^{iu(X_t - X_s)}e^{iw\mathbb{1}_B}] = e^{-u^2(t-s)/2}\mathbb{E}(e^{iw\mathbb{1}_B}), \quad \forall u, w \in \mathbb{R},$$

thus $X_t - X_s$ and $\mathbb{1}_B$ are independent for any $B \in \mathcal{F}_s$, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$.
Step 2. We will then prove (2.1) is true. Let $f(x) = e^{iux}$, then

$$f'(x) = iue^{iux}, \quad f''(x) = -u^2 e^{iux}.$$

By Itô's formula,

$$\begin{aligned} e^{iuX_t} - e^{iuX_s} = f(X_t) - f(X_s) &= \int_s^t f'(X_r)\,\mathrm{d}X_r + \frac{1}{2}\int_s^t f''(X_r)\,\mathrm{d}[X]_r \\ &= iu\int_s^t e^{iuX_r}\,\mathrm{d}X_r - \frac{u^2}{2}\int_s^t e^{iuX_r}\,\mathrm{d}r. \end{aligned} \tag{2.2}$$

Since $|e^{iuX_r}| = 1$, $e^{iuX_t} \in \mathcal{L}_A^2$, the first integral is a martingale by the property of Itô integral, thus

$$\mathbb{E}\left[\int_s^t e^{iuX_r}\,\mathrm{d}X_r \,\middle|\, \mathcal{F}_s\right] = 0.$$

Multiplying both sides of (2.2) by e^{-iuX_s} and take conditional expectation, we have

$$\mathbb{E}\left[e^{iu(X_t-X_s)} - 1 \,\middle|\, \mathcal{F}_s\right] = -\frac{u^2}{2}\mathbb{E}\left[\int_s^t e^{iu(X_r-X_s)}\,\mathrm{d}r \,\middle|\, \mathcal{F}_s\right].$$

Let

$$g(t) := \mathbb{E}[e^{iu(X_t-X_s)}|\mathcal{F}_s],$$

then applying Fubini's theorem for conditional expectations, we have

$$g(t) - 1 = -\frac{u^2}{2}\int_s^t g(r)\,\mathrm{d}r,$$

i.e.

$$g'(t) = -\frac{u^2}{2}g(t), \quad g(s) = 1.$$

Solving the ODE, we get the unique solution

$$g(t) = e^{-u^2(t-s)/2}. \qquad \square$$

Bibliography

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